Pure Strategy Dominance

By a theorem of Pearce for finite games:
a strategy is strictly dominated $\iff$ it is a never-best-reply.

However, the dominating strategy might be a mixed strategy.

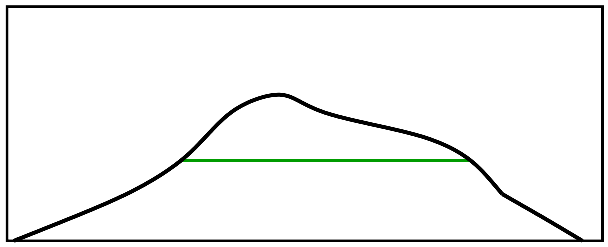
Player 2

		L	R
	t	1	1
Player 1	m	4	0
	b	0	4

Let us say that a strategy is *pure-undominated* if it is not dominated by a pure strategy.

- Payoffs in the matrix are for player 1. Strategy t is *not* pure-dominated, but is dominated by the mixed strategy in which player 1 plays m and b, each with probability $\frac{1}{2}$.
- Clearly, a strategy that is dominated cannot maximise expected utility under any belief.
- It is somewhat puzzling that mixed strategies enter the picture. Recall that for a mixed strategy that *does* maximize utility, each action in its support earns the same utility in expectation! That is, the agent is indifferent between all the pure strategies in the support. However, a strategy that dominates another strategy *need not* maximise expected utility (given some fixed belief).
- Dominance by pure strategies is often considered in the literature, either implicitly or explicitly. For instance, Apt [2] presents some nice results but primarily pertaining to pure strategy dominance.
- Pure strategy dominance seems to first have been considered by Börgers [3].

Own-Quasiconcave Games



Theorem:
If a game is own-quasiconcave, then:
strategy is never-best-reply $\iff$ dominated by a pure strategy.

Corollary 1:
Generalised result of Pearce for Infinite Games (cf. Zimper '06)

Corollary 2:
Set of rationalisable strategies R_i
$=$ set of serially pure-undominated strategies SPU_i

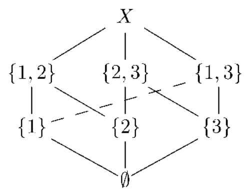
- A **game** is a tuple $(A_i, \pi_i)_{i \in N}$, where N is a set of agents; and for each $i \in N$, $A_i \subseteq \mathbf{R}^m$ is a (compact) set of actions; and $\pi_i : A_i \times A_{-i} \to \mathbf{R}$ player i's (suitably continuous) utility function.

- A game is **own-quasiconcave** if each A_i is convex, and each π_i is is quasiconcave on A_i for fixed $a_{-i} \in A_{-i}$.

- *Ad corollary 1*: Denote by $\triangle(S)$ the set of probability distributions over S. $\triangle(S)$ is convex. If we extend π to the expected utility function $\tilde{\pi}$ on $\triangle(A_i) \times \triangle(A_{-i})$, then $\tilde{\pi}_i$ is quasiconcave on $\triangle(A_i)$ for fixed a_{-i} (See also [1]).

Quasisupermodular Games

In quasisupermodular games each player's strategy space is a lattice (ordered by $\preceq$).



Best reply is $\preceq$-increasing in the strategies of the other players.

Serially undominated strategies must lie in $\preceq$-order-interval:
$I_i = \{\sigma \in \Sigma \mid \underline{\sigma} \preceq \sigma \preceq \overline{\sigma}\}$, that is $I_i \supseteq SPU_i$.

Now define a convexity notion based on $\preceq$. (follows H-LC 1996)

Say a function is $\Diamond$-quasiconcave iff upper level sets are $\Diamond$-convex.

Theorem. Let a game be own-$\Diamond$-quasiconcave, then $SPU_i = I_i$.

- Let $A_i \subseteq \mathbf{R}^m$ be a partially ordered by the pointwise ordering (denoted $\preceq$) for each $i \in N$. For finite subsets $F \subseteq A_i$, the pointwise sup and inf operations are well-defined and continuous. If closed under these operations, each A_i is a **topological lattice**. If closed under sup and inf for arbitrary subsets (not necessarily finite), the topological lattice is **complete**.

- For $a_i, a'_i \in A_i$, the **order-interval** $[a_i, a'_i]$ is given by the set $\{\tilde{a}_i \in A_i \mid a_i \preceq \tilde{a}_i \preceq a'_i\}$.

- The following convexity notion follows Horvath and Llinares Ciscar [4]. For a finite subset $F \subseteq A$, let $\Diamond(F) = \bigcup_{a_i \in F}[a_i, \sup F]$. A set $B \subseteq A_i$ is $\Diamond$-convex if for any finite subset $F \subseteq B$, $\Diamond(F) \in B$.

- We say that $f : A_i \to \mathbf{R}$ is **$\Diamond$-quasiconcave** if for each $c \in \mathbf{R}$, the upper level set $\{a_i \in A_i \mid f(a_i) > r\}$ is $\Diamond$-convex.

- *Remark* If A_i is a closed convex subset of the real line, then $B \subseteq A_i$ is $\Diamond$-convex iff for any two points $b_1, b_2 \in B$, the set $\{s \in S \mid b_1 \leq s \leq b_2\} \subseteq B$ (thus corresponding to the "ordinary" notion of convexity of a real line segment).

Additional Note: Quasisupermodular Games

We briefly introduce the class of quasisupermodular games (see Milgrom and Shannon [5]). Let each A_i be a complete lattice (for instance, $A_i \subseteq \mathbf{R}^m$, compact, and closed under sup and inf of all subsets $S \subseteq A_i$.). Endow $A := \prod_N A_i$ with the obvious product ordering. Then the function π_i is called quasisupermodular if for each $a \in A$ and $a' \in A$:

$$\pi_i(\inf\{a, a'\}) \leq \pi_i(a) \implies \pi_i(a') \leq \pi_i(\sup\{a, a'\})$$

$$\pi_i(\inf\{a, a'\}) < \pi_i(a) \implies \pi_i(a') < \pi_i(\sup\{a, a'\})$$

Definition 1: A quasisupermodular game is a game in which for each $i \in N$, A_i is a complete lattice, and π_i is quasisupermodular and (suitably) continuous on A.

Definition 2: A quasisupermodular game is own-$\Diamond$-quasiconcave if for each $i \in N$, A_i is a path-connected topological lattice, and π_i is $\Diamond$-quasiconcave on A_i for fixed a_{-i}.

Theorem. In a quasisupermodular, own-$\Diamond$-quasiconcave game, the set of serially purely-undominated strategies is an order interval $[\underline{a}_i, \overline{a}_i]$ and coincides with the set of rationalisable strategies.

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