Small Worlds and State Dependent Utility

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Jones is contemplating buying either a sedan, a convertible, or neither. His initial decision is one under certainty.

	Ω
Sedan (a_1)	$c_{1\Omega}$
Convertible (a_2)	$c_{2\Omega}$
Neither (a_3)	$c_{3\Omega}$

Jones must account for uncertain future possibilities in actually making his choice. Letting $V \subset \Omega$ denote the event that Jones manages to vacation, what was once a decision under certainty is now one under uncertainty.

	Ω	
	V	V^c
Sedan (a_1)	c_{1V}	c_{1V^c}
Convertible (a_2)	c_{2V}	c_{2V^c}
Neither (a_3)	c_{3V}	c_{3V^c}

Definition

Given a grand world decision problem $\langle \mathcal{A}(S), C, \mathbf{F}, \succ \rangle$ and any small world decision structure $\langle \mathcal{A}(\bar{S}), \bar{C}, \bar{\mathbf{F}}, \succ \rangle$ defined with respect to it, we say $\langle \mathcal{A}(\bar{S}), \bar{C}, \bar{\mathbf{F}}, \succ \rangle$ is a pseudo-microcosm if preferences over $\bar{\mathbf{F}}$ satisfy Savage's seven postulates.

Let $\bar{p}$ and p denote the probability measures derived from preferences over $\bar{\mathbf{F}}$ and $\mathbf{F}$ respectively. And for a given small world event $\bar{B} \subset \bar{S}$, let $[\bar{B}]$ denote the union of all elements of $\bar{B}$. Then $[\bar{B}] \subseteq S$.

Definition

Given a grand world decision problem $\langle \mathcal{A}(S), C, \mathbf{F}, \succ \rangle$ and any small-world $\langle \mathcal{A}(\bar{S}), \bar{C}, \bar{\mathbf{F}}, \succ \rangle$ defined with respect to it, we say $\langle \mathcal{A}(\bar{S}), \bar{C}, \bar{\mathbf{F}}, \succ \rangle$ is a microcosm if $\langle \mathcal{A}(\bar{S}), \bar{C}, \bar{\mathbf{F}}, \succ \rangle$ is a *pseudo-microcosm* such that $\bar{p}(\bar{B}) = p([\bar{B}])$ for all $\bar{B} \subseteq \bar{S}$.

Not every pseudo-microcosm is a microcosm. This is the problem of small worlds.

Let the grand world state space be $S = [(0,0),(1,0)]$ and let the grand world consequence set C={2,1,0}. To fix an algebra, let $\mathscr{B}(S)$ denote the set of all subsets of [(0,0),(1,0)] that are *finite* unions of subintervals of [(0,0),(1,0)], i.e. let $\mathscr{B}(S)$ denote the finite Borel Algebra on the unit interval. Then **F** is the set of all $\mathscr{B}(S)$-measurable functions **f** from $[(0,0),(1,0)]$ to $\{2,1,0\}$. Define $\succ$ on **F** as follows:

$$\mathbf{f} \succ \mathbf{g} \text{ iff } \int_S \mathbf{f}(x)dx > \int_S \mathbf{g}(x)dx \tag{1}$$

It follows that the *derived* probability measure p is the uniform Lebesgue measure on the unit interval, i.e.

$$p(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \textit{otherwise} \end{cases}$$

Let the "small world" be identical to the grand world in all respects except with $\bar{C} = \{\frac{3}{2}, 0\}$. Every small world consequence corresponds to a grand world act. Let $\frac{3}{2} \in \bar{C}$ correspond to the grand world act

$$d(x) := \begin{cases} 2 & 0 \leq x \leq \frac{1}{2} \\ 1 & \frac{1}{2} < x \leq 1 \end{cases}$$

It can be shown that the small world is a pseduo-microcosm and that the small world probability measure $\bar{p}$ is given thus:

$$\bar{p}(x) = \frac{d(x)}{\int_S d(x)dx} = \frac{2}{3}d(x) \qquad (2)$$

It follows that the small world probabilities for the events $A = [0, \frac{1}{2}]$ and $B = (\frac{1}{2}, 1]$. are as follows:

$$\bar{p}(A) = \frac{2}{3}\int_A d(s)ds = \frac{2}{3}; \bar{p}(B) = \frac{2}{3}\int_B d(s)ds = \frac{1}{3}$$

But $\bar{p}$ is not the uniform Lebesgue measure on the unit interval, and our pseudo-microcosm is not a microcosm.

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- Schervish, M., T. Seidenfeld, and J. Kadane. 1990, "State Dependent Utilities" Journal of the American Statistical Association 85: 840-7 reprinted in *Rethinking the Foundations of Statistics* Cambridge: Cambridge University Press, 1999, pp. 149-168